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Finite Element Analysis of Optical Waveguides

N. MABAYA, P. E. LAGASSE, AND P. VANDENBULCKE

***Abstract*—A finite element program for the analysis of anisotropic optical waveguides is described. The appearance of spurious numerical modes, due to the fact that the functional is nonpositive definite is discussed and a possible solution to the problem is presented. For isotropic waveguides it is shown that both *EH*- and *HE*-type modes can be very accurately approximated by two different scalar finite element programs. Finally, a method for calculating the attenuation of leaky modes in a single material integrated optic waveguide using this scalar finite element method is proposed.**

I. INTRODUCTION

IN THE field of optical communications monomode or quasi-monomode guides have become important due to the growing interest in single mode fiber and integrated optical waveguide structures. The analysis of such waveguides is not an easy problem since in general the geometry can be quite complicated and the materials anisotropic. The finite element method is probably the waveguide analysis method that is the most generally applicable and most versatile. Once a finite element program has been written any geometry and material combination that can be suitably represented by a division in triangles can be analyzed.

Ten years ago the finite element method was first used for the computation of the eigenmodes of dielectric loaded, conducting wall waveguides. More recently Yeh *et al.* [2], [3] have extended the use of the finite element method to the analysis of optical waveguides. Since [2] and [3] contain a very thorough description of the application of the finite element method to the analysis of optical fibers and integrated optic waveguides, this paper will only deal with four specific questions related to the use of this method:

1) the extension of the method to the case of anisotropic waveguides;
2) a discussion of the problems encountered when computing higher order modes;
3) the use of scalar functionals for computing the *HE* and *EH* modes of optical guides in the weakly guiding approximation. The accuracy of this method is discussed and the very important computational advantages of this approach are illustrated by a number of examples;
4) the extension of the finite element method to the analysis of the leaky modes of a single material optical waveguide by means of a boundary perturbation method.

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N Mabaya is with the University of Ghent, B-9000 Ghent, Belgium, on leave from the National University of Zaire, Kinshasa XI Zaire.
P. E. Lagasse and P. Vandenbulcke are with the University of Ghent, B-9000 Ghent, Belgium

II. ANISOTROPIC WAVEGUIDES

In integrated optical devices that contain electrooptic or elastooptic interactions, optical waveguides are made on crystal substrates, such as $LiNbO_3$. This means that a complete eigenmode analysis method has to be able to handle anisotropic guides. In Fig. 1 we consider the most general case: and anisotropic guiding region of arbitrary cross section and index variation and an anisotropic substrate region. If the crystal has a diagonal permittivity tensor one can rewrite Maxwell equations in terms of the longitudinal components E_z and H_z in the following way:

$$-\left[\frac{\partial}{\partial x}\left(A_x\epsilon_x\frac{\partial E_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_y\epsilon_y\left(\frac{\partial E_z}{\partial y}\right)\right)\right]$$
$$+\frac{\beta}{\omega}\left[\frac{\partial}{\partial y}\left(A_y\frac{\partial H_z}{\partial x}\right)-\frac{\partial}{\partial x}\left(A_x\frac{\partial H_z}{\partial y}\right)\right]=\epsilon_z E_z$$

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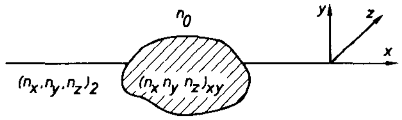

Fig. 1. General anisotropic optical waveguide.

$$\mu_0\left[\frac{\partial}{\partial x}\left(A_y\frac{\partial H_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_x\frac{\partial H_z}{\partial y}\right)\right] +\frac{\beta}{\omega}\left[\frac{\partial}{\partial y}\left(A_x\frac{\partial E_z}{\partial x}\right)-\frac{\partial}{\partial x}\left(A_y\frac{\partial E_z}{\partial y}\right)\right]=-\mu_0 H_z$$

where

$$A_x=\frac{1}{k_0^2\frac{\epsilon_x}{\epsilon_0}-\beta^2},\qquad A_y=\frac{1}{k_0^2\frac{\epsilon_y}{\epsilon_0}-\beta^2}$$

k_0 wavenumber in vacuum;
β wavenumber of the guided mode.

The finite element formulation is based on following variational expression for the previous equations:

$$\delta L=0$$

$$L=\int\int\tfrac{1}{2}\left[-\epsilon_z E_z^2-\mu_0 H_z^2+\epsilon_x A_x\left(\frac{\partial E_z}{\partial x}\right)^2 +\epsilon_y A_y\left(\frac{\partial E_z}{\partial y}\right)^2+\mu_0 A_y\left(\frac{\partial H_z}{\partial x}\right)^2+\mu_0 A_x\left(\frac{\partial H_z}{\partial y}\right)^2\right] +\frac{\beta}{\omega}\left[A_x\frac{\partial E_z}{\partial x}\cdot\frac{\partial H_z}{\partial y}-A_y\frac{\partial E_z}{\partial y}\cdot\frac{\partial H_z}{\partial x}\right]dS. \tag{1}$$

The natural boundary conditions for this function are

$$\bar{n}\bar{G}_E=0 \text{ and } \bar{n}\bar{G}_H=0$$

where

$$\bar{G}_E=\frac{\partial L}{\partial\left(\frac{\partial E_z}{\partial x}\right)}\bar{u}_x+\frac{\partial L}{\partial\left(\frac{\partial E_z}{\partial y}\right)}\bar{u}_y=-\frac{j}{\omega}(\bar{u}_z\times\bar{H}_{tr})$$

$$\bar{G}_H=\frac{\partial L}{\partial\left(\frac{\partial H_z}{\partial x}\right)}\bar{u}_x+\frac{\partial L}{\partial\left(\frac{\partial H_z}{\partial y}\right)}\bar{u}_y=\frac{j}{\omega}(\bar{u}_z\times\bar{E}_{tr}).$$

This means that the natural boundary conditions of functional (1) are the continuity of the tangential components along the discontinuities of the transversal electric and magnetic field. If the waveguide is isotropic, (1) is reduced to the well-known functional discussed, for example, in [1]–[3]. Starting from (1) a general finite element program for the analysis of anisotropic optical waveguides has been written [4]. When using such a program one is faced with a number of problems and trade offs.

1) The choice of the type of elements and the number of elements needed to model the waveguide. The most simple triangular element assumes a linear interpolation between the field values at the corner points of the triangle. Using this type of element one obtains large but sparse matrix equations. By careful numbering of the nodal points, band matrices can be obtained. Instead of the linear elements, one can also use triangular elements with higher order polynomial interpolation functions. The drawback is that the programming effort for those higher order elements is quite large. The advantage is that one can obtain accurate results with much smaller matrix dimensions. We have found for example that in the case of a rectangular overlay guide, where a small number of triangles is sufficient to model the geometry, one obtains about the same accuracy with 900 linear elements or 928 nodal points as with only 9 fourth-order elements or 87 nodal points. Since the geometry or index variation of some guides can be so complicated, as to require a large number of triangles, the finite element program allows one to specify the desired element order between 1 and 4.

2) The modeling of the infinite transverse extent of the waveguide always represents a problem. Three possible solutions are: a) imposing an artificial zero boundary condition for E_z and H_z at a large enough distance from the guide; b) use sector elements [3] that assume some exponential decay for the field; or c) implementing the radiation condition through an integral equation at the boundary of the finite element region. The last method, although exact, leads to such a complicated set of equations, that it is numerically impractical to use. The sector elements would be ideal if one could find the exponential decay factor, as a result of the variational process. Since this leads to nonlinear equations one has to determine the best exponential decay by trial and error for each point on the dispersion characteristic. The first method has as advantage its simplicity. It has been used for the calculation of the results presented in this paper but care has been taken to make sure that the influence of the position of this zero boundary condition on the obtained results, was negligible.

3) The most serious difficulty in using the finite element analysis, for open dielectric waveguides, is the appearance of spurious, nonphysical modes. This means that a number of the eigenvalues and eigenvectors of the matrix eigenvalue problem, do not represent physical modes of the waveguide, but are spurious results introduced by the numerical technique. The reason for the appearance of the spurious modes is probably the fact that the functional (1) is not positive definite since A_x or A_y can be positive or negative, depending whether the element is in the guide or in the substrate [1], [5], [6]. If one is interested only in the calculation of the lowest propagating mode, the appearance of those spurious modes is not much of a problem. The lowest order mode usually corresponds to the first positive eigenvalue of the matrix equation. This can easily be checked by plotting the calculated field values. In case of a nonphysical mode the fields vary in a random fashion over the guide cross section. If one wants to compute a set of higher order modes, it becomes more difficult and very cumbersome to distinguish between the spurious and the physical modes of the guide.

However, we have found that by explicitly enforcing the

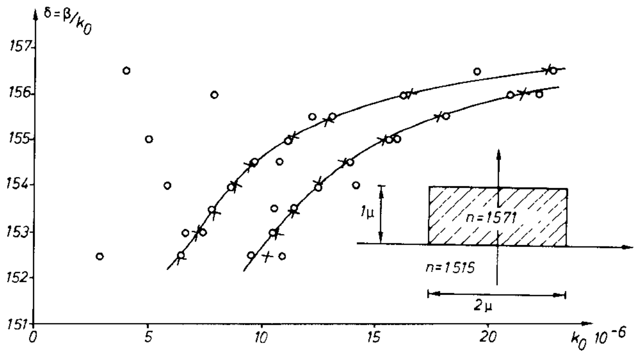

Fig. 2. Dispersion characteristic of a rectangular overlay guide—occurence of spurious modes.

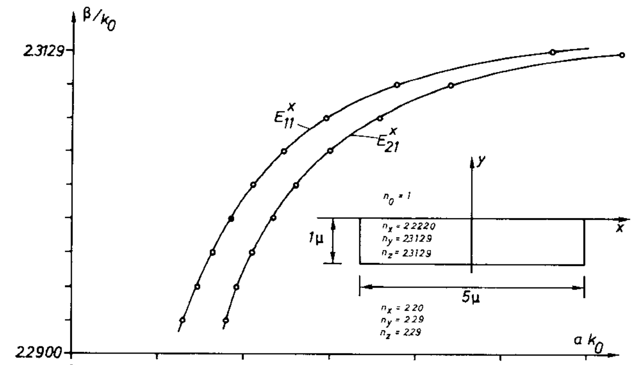

Fig. 3. Dispersion characteristic for the E_{11}^x and E_{21}^x mode of a LiNbO$_3$ waveguide.

continuity of the tangential components of the transversal fields, at the interfaces, by means of Lagrange multipliers, most of the spurious modes disappear. Since this boundary condition is in principle already enforced by the natural boundary conditions of the functional (1), we have no firm mathematical proof for this method. The effectiveness of the technique is illustrated by an example shown in Fig. 2. The circles represent all the solutions of the classic finite element program, while the results of the finite element program with continuity conditions are indicated by crosses. One can clearly see that all the results of this last program, lie on the dispersion characteristic of the modes of the guide. The ability to calculate a set of modes of an anisotropic waveguide is illustrated in Fig. 3 by a plot of the dispersion characteristic of the E_{11}^x and E_{21}^x modes of a guide made on a Y-cut LiNbO$_3$ substrate. Fig. 4 shows the contour lines for E_z and H_z for those two modes. The disadvantage of this method lies in the greatly increased complexity of the program and of the numerical operations that have to be done for enforcing those continuity conditions. For very complicated guide geometries for example the accumulation of rounding errors becomes a problem. If the guide is isotropic or if it can be approximated by an equivalent isotropic guide, we propose in the next paragraph an approximate finite element formulation that allows a much easier and faster calculation of the different modes of the guide.

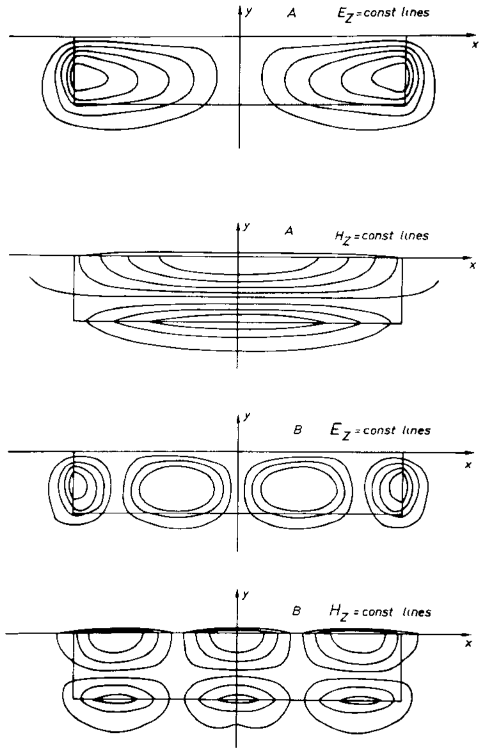

Fig. 4. Contour lines for E_z and H_z for the E_{11}^x and E_{21}^x mode of the LiNbO$_3$ waveguide shown in Fig. 3.

III. Approximate Scalar Finite Element Formulations

If the optical guide is isotropic, we propose two different scalar formulations, that yield excellent approximations for the EH and HE type of mode of an integrated optical waveguide. As an example we consider a rectangular overlay waveguide with height a and width $2a$. The refractive index of the guide and of the substrate is, respectively, 1.5 and 1.45. Using the vectorial finite element program described in previous paragraph and a division consisting of 9 fourth-order triangular elements, we find following points of the dispersion characteristic of the lowest mode:

HE mode: $k_0a = 11.4512$

$\beta a = 16.9478$

EH mode: $k_0a = 11.7012$

$\beta a = 17.3178.$

The scalar approximation for the HE modes is based on the following functional:

$$L = \int\int \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 - k_0^2 n^2 \phi^2 + \beta^2 \phi^2 \right] dS. \quad (2)$$

This functional has the continuity of $\partial\phi/\partial n$ as natural boundary condition. A finite element program based on functional (2) yields β as the eigenvalue of the matrix equation for a given k_0. In the case of an infinite slab guide equation (2) gives an exact variational expression for the TE slab modes. If we consider again the rectangular overlay guide one finds following points of the dispersion characteristic:

$$k_0 a = 11.4512$$
$$\beta a = 16.9481.$$

This is almost identical to the result obtained for the *HE* mode with the full vectorial analysis.

The scalar approximation for the *EH* modes is based on the following functional:

$$L = \iint \left[\frac{1}{n^2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{n^2}\left(\frac{\partial\phi}{\partial y}\right)^2 + \frac{\beta^2}{n^2}\phi^2 - k_0^2\phi^2 \right] dS. \quad (3)$$

This functional has the continuity of $((1/n^2)(\partial\phi/\partial n))$ as natural boundary condition. A finite element program based on this functional yields k_0 as eigenvalue of the matrix equation for a given β. In the case of an infinite slab guide, (3) gives an exact variational expression for the TM slab modes. Considering again the previous rectangular overlay guide one finds following points on the dispersion characteristic:

$$k_0 a = 11.7086$$
$$\beta a = 17.3178.$$

This is almost identical to the result obtained for the *EH* modes with the full vectorial analysis.

From those examples one can see that the two scalar finite element formulations form an excellent approximation for the *HE*- and *EH*-type modes of the optical waveguide, even in the case where the width to height ratio of the guide is small. The main advantages of this scalar approximation are as follows.

1) The dimensions of the matrices are reduced by a factor of 2 which means a reduction of the computer time by approximately a factor of 4.

2) The two scalar functionals are positive definite (or can immediately be made positive definite). All the eigenvalues are, therefore, positive and each one corresponds to a physical mode of the guide. This means that one can now easily compute the higher order modes of the guide.

To illustrate the use of those scalar finite element approximation a number of modes of two different waveguides have been calculated. First we consider again the rectangular overlay guide described earlier. In Fig. 5 contour plots for the scalar field ϕ are shown for the 5 lowest order *HE* modes of the guide. The normalized wavenumber $k_0 a$ for all modes is equal to 25. For a lower value of $k_0 a$ such as 12.5 only two modes are propagated. As can be seen in Fig. 6 the field extends further into the substrate when the mode is close to cutoff. It is easier to calculate modes close to cutoff with this scalar finite element method since it is possible to use more triangles to model the substrate for a given maximum matrix size.

As a second example we consider the trapezoidal overlay guide [7], [8] shown in Fig. 7. Such a guide is obtained when the guide material is not completely etched away, so that a thin layer remains over the complete surface of the waveguide. One can see how the four lowest order modes

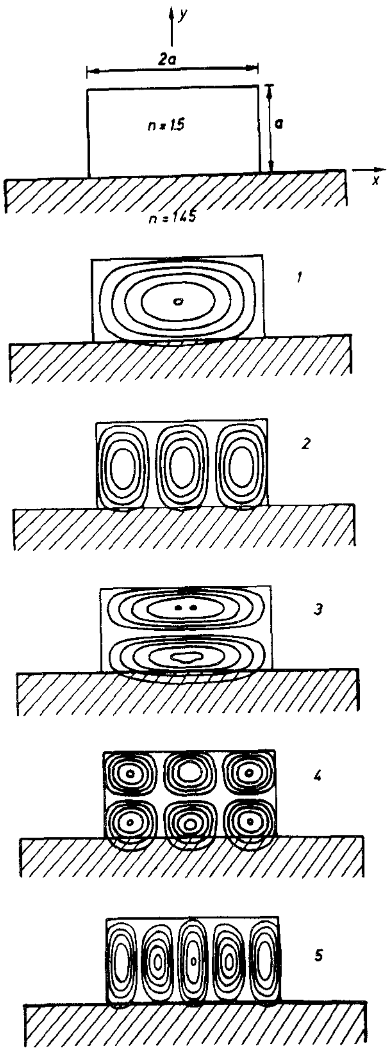

Fig. 5. Contour lines for the scalar field ϕ of the five lowest order modes of a rectangular overlay guide as calculated by the scalar finite element program ($k_0 a = 25$).

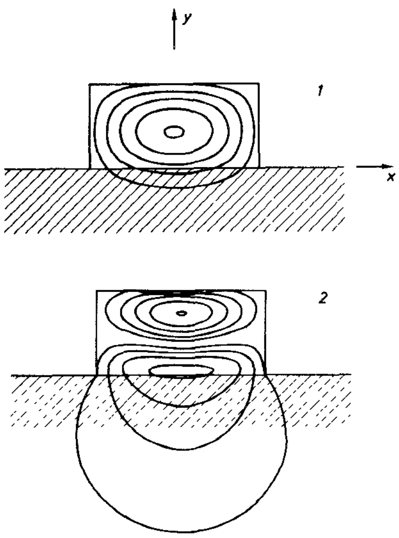

Fig. 6. Contour lines for the scalar field ϕ of the two lowest order modes of the rectangular overlay guide for $k_0 a = 12.5$. Second mode is close to cutoff.

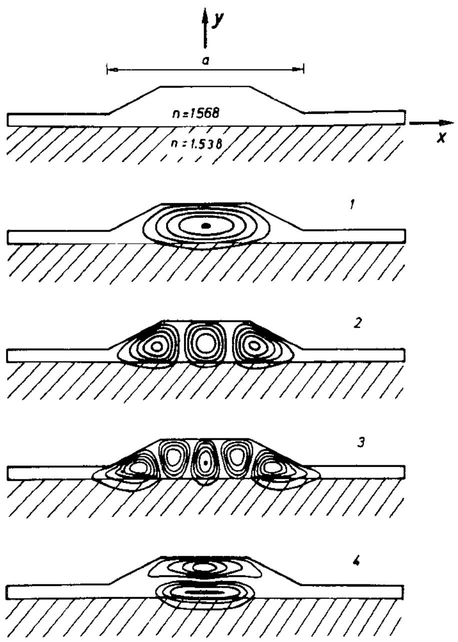

Fig. 7. Contour lines for the scalar field ϕ of the four lowest order modes of trapezoidal guide.

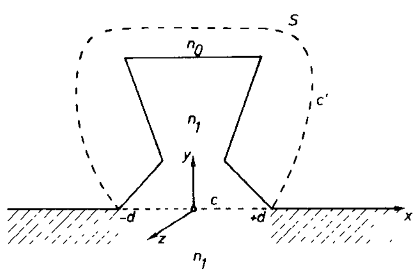

Fig. 8. Cross section of single material "waisted-rib" optical waveguide.

of such a guide are easily computed using only 12 fourth-order elements.

IV. Leaky Modes in Single Material Guides

Recently a single material "waisted rib" optical waveguide defined by means of preferential etching of a GaAs substrate has been described [9]. A typical cross section of such a guide is sketched in Fig. 8. Since the guide has the same refractive index as the substrate a pure guided mode cannot exist, so that the light coupled in such a waveguide will leak into the substrate. If the attenuation of such a leaky mode is low enough, the guide can still be used in integrated optic structures. In order to design a guide with sufficiently low leakage one needs a method for computing the attenuation of the leaky modes. The finite element method with an artificial zero field boundary condition obviously cannot be used for the direct calculation of leaky modes. For the case of low leakage we propose a perturbation method based on the scalar finite element method.

We consider the single material topographic optical waveguide shown in Fig. 8. If we assume the field to be zero on the boundaries c and c' we can compute the eigenmodes of this guide by means of the scalar finite element method. A guided mode ϕ_1 of this waveguide obeys the wave equation

$$\nabla^2\phi_1 + k^2\phi_1 = 0 \tag{4}$$

with as boundary condition: $\phi_1 = 0$ on c and c'. If the propagation constant of this mode is k_1 we have

$$\phi_1(x,y,z) = \Phi_1(x,y)\cdot e^{-jk_1 z}. \tag{5}$$

Due to the presence of the substrate, the "mode" ϕ_2 of the real guide is leaky

$$\phi_2(x,y,z) = \Phi_2(x,y)\cdot e^{-jk_2 z}e^{-\alpha z} \tag{6}$$

with

$$\nabla^2\phi_2 + k^2\phi_2 = 0 \tag{7}$$

where ϕ_2 satisfies the radiation condition at infinity in the substrate. In order to apply the boundary perturbation method we derive first an impedance relation

$$\Phi_2 = F\left(\frac{\partial \Phi_2}{\partial n}\right)$$

for Φ_2 on the boundary c. Using Greens function for two-dimensional halfspace, one finds

$$\Phi_2(x_0) = -\frac{1}{2j}\int_{-d}^{+d} H_0^{(1)}[\gamma|x-x_0|]\frac{\partial \Phi_2}{\partial y}\cdot dx \tag{8}$$

where

$$\gamma = \sqrt{k^2 - k_2^2}$$

and $H_0^{(1)}$ is the zeroth-order Hankel function of the first kind.

Equations (4) and (7) yield after integration over the guide cross section S

$$\int\!\!\int_S \left[\phi_2^\star \nabla^2\phi_1 - \phi_1\nabla^2\phi_2^\star\right] dS = 0.$$

After the application of Greens theorem, the substitution of (5), (6) and taking the boundary conditions into account, one finds

$$\int_c \Phi_2^\star \frac{\partial \Phi_1}{\partial y}\cdot dc = \int\!\!\int_S \Phi_1\Phi_2^\star\left[k_1^2 + (jk_2 - \alpha)^2\right] dS.$$

Using (8) and approximating Φ_2 by Φ_1, one finally obtains

$$\alpha = -\frac{\int_{-d}^{+d}\left\{\int_{-d}^{+d} J_0[\gamma|x-x_0|]\frac{\partial \Phi_1}{\partial y}\cdot dx\right\}\frac{\partial \Phi_1}{\partial y}dx_0}{4k_1\int\!\!\int_S |\Phi_1|^2\, dS}. \tag{9}$$

This formula allows one to calculate the attenuation coefficient α, once the mode Φ_1 of the truncated guide has been computed by means of the finite element method.

In Fig. 9 the contour lines for the fundamental mode of the guide with a zero boundary condition on c are plotted for $k_0 = 1\ \mu\text{m}^{-1}$ and $k_0 = 3\ \mu\text{m}^{-1}$. Application of formula (9) yields an attenuation of about 100 dB/mm for $k_0 = 1\ \mu\text{m}^{-1}$ and 14 dB/mm for $k_0 = 3.5\ \mu\text{m}^{-1}$. Although still high, this last value indicates that better geometries and modes probably can be found, so that acceptable values for the attenuation are obtained. Even if this perturbation

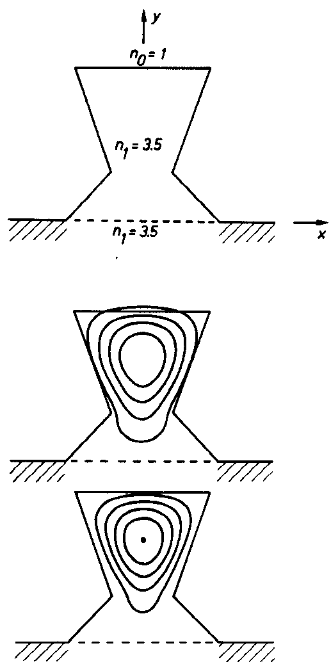

Fig. 9. Contour lines for the scalar field ϕ of the fundamental mode of GaAs "waisted-rib" waveguide for (a) $k_0=1$ μm^{-1} and (b) $k_0=3$ μm^{-1}.

method is not highly accurate, we hope to use it to find out which guide cross sections have any chance of providing relatively low loss light guidance.

V. Conclusions

In this paper several finite element programs for the computation of the guided modes of optical waveguides have been discussed. First a very general program for the analysis of anisotropic guides was presented. The advantages and limitations of this program have been described. A possible solution to the problem of the spurious numerical modes, encountered when calculating higher order modes, has been proposed. In the case of isotropic waveguides, it was shown that two scalar finite element formulations can provide very accurate solutions to the eigenmode problem. This approach has as main advantages: the smaller matrix dimensions, less computer time, no spurious modes and the capability of easily computing higher order modes. Finally a boundary perturbation method has been outlined that allows one to calculate the attenuation coefficient of leaky modes in single material guides, starting from a finite element calculation.

Acknowledgment

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